

Toric Sasaki-Einstein manifolds and Heun equations

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Abstract

Symplectic potentials are presented for a wide class of five dimensional toric Sasaki-Einstein manifolds, including $L^{a,b,c}$ which was recently constructed by Cvetic et al. The spectrum of the scalar Laplacian on $L^{a,b,c}$ is also studied. The eigenvalue problem leads to two Heun's differential equations and the exponents at regular singularities are directly related to the toric data. By combining knowledge of the explicit symplectic potential and the exponents, we show that the ground states, or equivalently holomorphic functions, have one-to-one correspondence with the integral lattice points in the convex polyhedral cone. The scaling dimensions of the holomorphic functions are simply given by the scalar products of the Reeb vector and the integral vectors, which are consistent with R -charges of the BPS states in the dual quiver gauge theories.

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1 Introduction

Toric Sasaki-Einstein manifolds have been attracted much attention. Five dimensional toric Sasaki-Einstein manifolds X_5 can be used especially as the type IIB backgrounds whose near horizon geometries have the form $AdS_5 \times X_5$. Recently, nontrivial infinite families of toric Sasaki-Einstein manifolds were explicitly constructed [1, 2, 3, 4] and many new insights [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] were obtained for the AdS/CFT correspondence [20]. CFT duals to the toric Sasaki-Einstein manifolds are $\mathcal{N} = 1$ superconformal quiver gauge theories in $3 + 1$ dimensions [21, 22, 23]. In these cases, both the geometry and the gauge theory can be characterised by a common data, called a toric data. On the CFT side, the formalism to determine the quiver gauge theory from the toric data is now well established and the extracted data can be beautifully encoded in diagrams on a torus, called the brane tilings [24, 25, 17, 26]. However, on the gravity side, the procedure to obtain the metric from the toric data is still not established.

A Sasaki-Einstein manifold is a Riemannian manifold whose metric cone is Ricci-flat Kähler i.e. Calabi-Yau. So far, many works on Calabi-Yau manifolds have been done using the complex coordinates. However, Calabi-Yau or Kähler manifolds are also symplectic. As is discussed in [27], a real symplectic viewpoint can be useful in the Kähler toric geometry: for example, explicit construction of extremal Kähler metrics and spectral properties of the toric manifolds. Therefore, some aspects of the toric Sasaki-Einstein manifolds may become transparent in the symplectic coordinates [9]. The key object in the symplectic approach is the so called symplectic potential, which is not well studied object compared to the Kähler potential. In the first part of this paper, we study the symplectic potentials for various toric Sasaki-Einstein manifolds.

Then, we analyse the spectrum of the scalar Laplacian for the toric Sasaki-Einstein manifolds $L^{a,b,c}$ [3, 4] since the four dimensional base spaces of $L^{a,b,c}$ are the most general *orthotoric* Kähler-Einstein spaces [28, 10]. We will show that the eigenvalue problem leads to two Heun's differential equations [29], and the exponents at the regular singularities are characterised by the toric data. Although there is some progress in the construction of Heun functions (see, for example, [30]), it is still difficult to solve Heun's differential equations. The determination of the spectrum of $L^{a,b,c}$ is also interesting as an eigenvalue problem of Heun's. In this paper, we obtain some polynomial solutions, which correspond to the ground states and the first excited states.

Due to the AdS/CFT correspondence, there must be pairings between the ground states and the BPS operators in the dual quiver theories. By combining knowledge of the

symplectic potential and the connection between the exponents and the toric data, the corresponding BPS operators can be naturally identified.

This paper is organised as follows. In the next section, after a brief review of the symplectic approach to the toric Sasaki-Einstein manifolds, we present the symplectic potentials for various cases. In section 3, we show how two Heun's differential equations appear in the eigenvalue problem of the scalar Laplacians. In section 4, some polynomial type Heun functions are obtained. Relations between the ground states, holomorphic functions, and the BPS mesonic operators are argued in section 5. Section 6 is devoted to discussions. Appendix treats some details on the symplectic potential for $L^{a,b,c}$.

2 Toric Sasaki-Einstein manifolds

2.1 Review of symplectic approach

In this subsection, we briefly review the symplectic approach to the toric Sasaki-Einstein manifolds according to [9]. Let X_5 be a five dimensional compact Riemannian manifold with a metric g . The cone metric on $C(X_5) = X_5 \times \mathbb{R}_+$ is given by $g_{C(X_5)} = dr^2 + r^2 g$ ($r > 0$). Sasaki-Einstein manifolds can be defined as compact Einstein spaces whose metric cones are Ricci-flat Kähler (i.e. Calabi-Yau). The cone of a Sasaki-Einstein manifold is also regarded as a symplectic manifold with the symplectic form

$$\omega = d\left(\frac{r^2}{2}\eta\right) = r dr \wedge \eta + \frac{r^2}{2} d\eta. \quad (2.1)$$

Here η is the contact one-form which is dual to the distinguished Killing vector B , called the Reeb vector. The symplectic coordinates $z = (\xi^1, \xi^2, \xi^3, \phi_1, \phi_2, \phi_3)$ on $C(X_5)$ can be chosen such that

$$(r^2/2)\eta = \xi^i d\phi_i =: \xi. \quad (2.2)$$

The one-form ξ defined here is related to the symplectic form as $\omega = d\xi = d\xi^i \wedge d\phi_i$.

A five dimensional toric Sasaki-Einstein manifold is a Sasaki-Einstein manifold with three $U(1)$ isometries : $G = T^3$. The isometry group G acts as translations for the coordinates ϕ_i . Let \mathfrak{g} be its Lie algebra and \mathfrak{g}^* the dual vector space of \mathfrak{g} . The generators of \mathfrak{g} are $\partial/\partial\phi_i$ ($i = 1, 2, 3$). The moment map for the T^3 -action is given by

$$\begin{aligned} \Phi : C(X_5) &\rightarrow \mathfrak{g}^* \cong \mathbb{R}^3, \\ z &\mapsto \xi = (\xi^1, \xi^2, \xi^3). \end{aligned} \quad (2.3)$$

The image of the moment map is a convex polyhedral cone:

$$\Phi(C(X_5)) = \{ \xi \in \mathfrak{g}^* \mid \langle v_A, \xi \rangle \geq 0, \quad A = 1, 2, \dots, D \}, \quad (2.4)$$

where

$$v_A = ((v_A)_1, (v_A)_2, (v_A)_3) = (v_A)_i \frac{\partial}{\partial \phi_i}, \quad (v_A)_i \in \mathbb{Z}. \quad (2.5)$$

The pairing between the vector v_A and the one form ξ is given by $\langle v_A, \xi \rangle = (v_A)_i \xi^i$. Calabi-Yau conditions require that the vectors v_A ($A = 1, 2, \dots, D$) lie on a plane in the vector space \mathfrak{g} . By appropriate choice of basis, the first component of these vectors can be set to be one: $v_A = (1, w_A)$ with $w_A \in \mathbb{Z}^2$. The set of vectors $\{v_A\}$ is called a toric data. Each subspace with $\langle v_A, \xi \rangle = 0$ ($A = 1, 2, \dots, D$) is called a facet. Note that the Sasaki-Einstein manifold is a T^3 -fibration over the hypersurface $\langle B, \xi \rangle = 1/2$ in the polyhedral cone.

In the symplectic coordinates, the cone metric of the toric Sasaki-Einstein manifold can be expressed as

$$g_{C(X_5)} = G_{ij}(\xi) d\xi^i d\xi^j + G^{ij}(\xi) d\phi_i d\phi_j, \quad (2.6)$$

$$G_{ij} G^{jk} = \delta_i^k. \quad (2.7)$$

The metric is a cone if and only if G_{ij} is homogeneous degree -1 in ξ^k . The Kähler condition implies that G_{ij} can be expressed by using a symplectic potential G :

$$G_{ij}(\xi) = \frac{\partial^2 G(\xi)}{\partial \xi^i \partial \xi^j}. \quad (2.8)$$

The direct problem, to obtain the symplectic potential and the toric data from the metric, is not so formidable task. On the other hand, the inverse problem, to obtain the Sasaki-Einstein metric from the toric data, is very difficult.

Suppose we only know a toric data and do not know the corresponding Sasaki-Einstein metric. In the symplectic approach, we should determine the symplectic potential. The moduli space of the symplectic potentials for the toric Sasakian manifolds¹ can be written as $\mathcal{C}_0 \times \mathcal{H}(1)$. Here \mathcal{C}_0 is the interior of the dual polyhedral cone and it is identified with the space of the Reeb vectors $B = (B_1, B_2, B_3)$. Also, $\mathcal{H}(1)$ is the space of smooth homogeneous degree one functions h on the polyhedral cone. The homogeneous one condition is required to guarantee that G_{ij} is homogeneous degree -1 .

¹Sasakian manifolds can be defined as compact spaces whose metric cones are Kähler.

We must determine two quantities B and h in order to obtain the Sasaki-Einstein metric. There is a well-established procedure to determine the Reeb vector B from the toric data. It is the Z -minimisation. This procedure is independent of the specification of h . For any toric data, the first component of the Reeb vector is fixed to be three: $B_1 = 3$.

The symplectic potential G can be written as follows

$$G = G^{\text{can}} + G^B + h, \quad (2.9)$$

where

$$G^{\text{can}} = \sum_{A=1}^D \frac{1}{2} \langle v_A, \xi \rangle \log \langle v_A, \xi \rangle, \quad (2.10)$$

$$G^B = \frac{1}{2} \langle B, \xi \rangle \log \langle B, \xi \rangle - \frac{1}{2} \langle B^{\text{can}}, \xi \rangle \log \langle B^{\text{can}}, \xi \rangle, \quad (2.11)$$

$$B^{\text{can}} := \sum_{A=1}^D v_A. \quad (2.12)$$

The canonical part G^{can} , which is completely fixed by the toric data, specifies the singular behaviour of G at the facets. By adding the second term G^B , the Reeb vector shifts from B^{can} to B . The third term $h \in \mathcal{H}(1)$ must be regular at the facets.

The Ricci-flatness condition is given by the Monge-Ampère equation

$$\det \left(\frac{\partial^2 G}{\partial \xi^i \partial \xi^j} \right) = \text{const} \times \exp \left(2\gamma^i \frac{\partial G}{\partial \xi^i} \right). \quad (2.13)$$

Here γ^i are constants. In order that the corresponding metric is smooth, the constant vector $\gamma = (\gamma^i)$ must be chosen as [27]

$$\gamma = (-1, 0, 0). \quad (2.14)$$

In general, $G^{\text{can}} + G^B$ is not a solution to the Monge-Ampère equation. So the discrepancy part h is necessary. In order to obtain the toric Sasaki-Einstein metric, we should solve this quite non-linear partial differential equation for h . In the subsequent subsections, several solutions to the Monge-Ampère equations are presented.

2.2 $C(T^{1,1})$ and $C(T^{1,1}/\mathbb{Z}_2)$

The explicit homogeneous $T^{1,1}$ metric was constructed in [31]. The $T^{1,1}$ case is the first nontrivial example of toric Sasaki-Einstein/quiver duality [22].

The toric data for $T^{1,1}$ is given by [23, 5]

$$v_1 = (1, 1, 1), \quad v_2 = (1, 0, 1), \quad v_3 = (1, 0, 0), \quad v_4 = (1, 1, 0). \quad (2.15)$$

The cone $C(T^{1,1})$ is the conifold. By the Z -minimisation, the Reeb vector is determined as $B = (3, 3/2, 3/2)$ [9] and $B^{\text{can}} = \sum_{A=1}^4 v_A = (4, 2, 2)$.

The toric data for $T^{1,1}/\mathbb{Z}_2$ is given by [23]

$$v_1 = (1, 0, 1), \quad v_2 = (1, 1, 0), \quad v_3 = (1, 2, 1), \quad v_4 = (1, 1, 2). \quad (2.16)$$

The cone $C(T^{1,1}/\mathbb{Z}_2)$ in this case corresponds to the complex cone over the zeroth Hirzebruch surface \mathbb{F}_0 . The Reeb vector is determined as $B = (3, 3, 3)$ and $B^{\text{can}} = \sum_{A=1}^4 v_A = (4, 4, 4)$.

For $C(T^{1,1})$ or $C(T^{1,1}/\mathbb{Z}_2)$, the canonically constructed symplectic potential $G = G^{\text{can}} + G^B$ itself is a solution to the Monge-Ampère equation. One can check that it indeed satisfies the Monge-Ampère equation:

$$\det \left(\frac{\partial^2 G}{\partial \xi^i \partial \xi^j} \right) = \mathcal{C} \frac{\langle B, \xi \rangle}{\langle v_1, \xi \rangle \langle v_2, \xi \rangle \langle v_3, \xi \rangle \langle v_4, \xi \rangle} = \text{const} \times \exp \left(-2 \frac{\partial G}{\partial \xi^1} \right). \quad (2.17)$$

Here the constant \mathcal{C} is $1/8$ for $C(T^{1,1})$ and $1/2$ for $C(T^{1,1}/\mathbb{Z}_2)$. The equation above is also valid for other choices of the toric data.

2.3 Symplectic potentials of $C(Y^{p,q})$

An infinite family of Sasaki-Einstein metrics with cohomogeneity one, called $Y^{p,q}$, was obtained in [1]².

The toric data of $Y^{p,q}$ is given by [5]

$$v_1 = (1, -1, -p), \quad v_2 = (1, 0, 0), \quad v_3 = (1, -1, 0), \quad v_4 = (1, -2, -p + q). \quad (2.18)$$

From the Z -minimisation, the Reeb vector is determined as

$$B = \left(3, -3, -\frac{3}{2} \left(p - q + \frac{1}{3\ell} \right) \right), \quad (2.19)$$

where

$$\ell = \frac{q}{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}}. \quad (2.20)$$

²The positive integers p and q are assumed to be relatively prime. This ensures that $Y^{p,q}$ is simply-connected, and thus diffeomorphic to $S^2 \times S^3$.

We find that a homogeneous regular solution is given by

$$h = \frac{1}{2} \langle B, \xi \rangle \log \frac{\langle v_5, \xi \rangle}{\langle B, \xi \rangle} + \frac{1}{2} \sum_{A=1,3} \langle v_A, \xi \rangle \log \frac{\langle B^{\text{can}}, \xi \rangle}{\langle v_5, \xi \rangle} + \frac{1}{2} \sum_{A=2,4} \langle v_A, \xi \rangle \log \frac{\langle B^{\text{can}}, \xi \rangle}{|\langle v_6, \xi \rangle|}, \quad (2.21)$$

where additional vectors v_5 and v_6 are defined by $v_5 := B - v_1 - v_3$, $v_6 := -v_2 - v_4$, and $B^{\text{can}} = \sum_{A=1}^4 v_A$. The symplectic potential of $C(Y^{p,q})$ is summed up into a simple form

$$\begin{aligned} G(\xi) &= G^{\text{can}} + G^B + h \\ &= \frac{1}{2} \sum_{I=1}^6 \langle v_I, \xi \rangle \log |\langle v_I, \xi \rangle|. \end{aligned} \quad (2.22)$$

One can check that the symplectic potential (2.22) indeed satisfies the Monge-Ampère equation:

$$\det \left(\frac{\partial^2 G}{\partial \xi^i \partial \xi^j} \right) = \frac{1}{32al^2} \frac{\langle v_6, \xi \rangle^2}{\langle v_1, \xi \rangle \langle v_2, \xi \rangle \langle v_3, \xi \rangle \langle v_4, \xi \rangle \langle v_5, \xi \rangle} = \text{const} \times \exp \left(-2 \frac{\partial G}{\partial \xi^1} \right), \quad (2.23)$$

where

$$a = \frac{1}{2} - \frac{(p^2 - 3q^2)}{4p^3} \sqrt{4p^2 - 3q^2}. \quad (2.24)$$

This is one of our main results. The expression of the symplectic potential (2.22) and the Monge-Ampère relation (2.23) are valid not only for the choice (2.18) but also for other choices of the toric data of $Y^{p,q}$ [7, 8], which are related to (2.18) by translations and $SL(2, \mathbb{Z})$ rotations.

2.4 Symplectic potentials of $C(L^{a,b,c})$

Cohomogeneity two generalisation of the $Y^{p,q}$ metric, called the $L^{a,b,c}$ metric, was constructed in [3, 4].

The toric data of $L^{a,b,c}$ is given by [17]

$$v_1 = (1, 1, 0), \quad v_2 = (1, ak, b), \quad v_3 = (1, -al, c), \quad v_4 = (1, 0, 0), \quad (2.25)$$

where the integers k, l are chosen such that $ck + bl = 1$. It holds that $av_1 - cv_2 + bv_3 - dv_4 = 0$ with $d = a + b - c$.

The Reeb vector B can be obtained by using a solution of certain quartic equation [16]. But, it is difficult to write B in a concise form. We do not write it because the explicit form of the components B_i is not relevant for solving the Monge-Ampère equation. For an implicit expression, see (3.13).

We solved the Monge-Ampère equation for $L^{a,b,c}$ cases. The solution is summarised as follows: the symplectic potential for $C(L^{a,b,c})$ can be written as

$$G(\xi) = \frac{1}{2}\langle B, \xi \rangle \log \langle B, \xi \rangle + \frac{1}{2} \sum_{m=1}^3 \langle v_{2m-1}, \xi \rangle \log |x - x_m| + \frac{1}{2} \sum_{m=1}^3 \langle v_{2m}, \xi \rangle \log |y - y_m|, \quad (2.26)$$

where two auxiliary vectors in \mathfrak{g} are introduced as

$$v_5 := B - v_1 - v_3, \quad v_6 := B - v_2 - v_4. \quad (2.27)$$

Here x and y are functions of ξ^i and are defined implicitly by

$$\langle v_2, \xi \rangle = \frac{r^2}{4\alpha}(\alpha - x)(1 - y), \quad \langle v_4, \xi \rangle = \frac{r^2}{4\beta}(\beta - x)(1 + y), \quad \langle B, \xi \rangle = r^2/2. \quad (2.28)$$

Note that

$$\langle v_6, \xi \rangle = \frac{r^2}{4\alpha\beta}x(\alpha + \beta + (\alpha - \beta)y). \quad (2.29)$$

The constants α, β, x_i, y_i ($i = 1, 2, 3$) obey certain consistency relations. One can set the constants y_i as follows:

$$y_1 = 1, \quad y_2 = -1, \quad y_3 = \frac{\beta + \alpha}{\beta - \alpha}. \quad (2.30)$$

The following relations should be hold:

$$\begin{aligned} \langle v_1, \xi \rangle &= \frac{r^2}{4} \frac{(\alpha + \beta + (\alpha - \beta)y - 2x_1)}{(x_1 - x_2)(x_1 - x_3)}(x - x_1), \\ \langle v_3, \xi \rangle &= \frac{r^2}{4} \frac{(\alpha + \beta + (\alpha - \beta)y - 2x_2)}{(x_2 - x_1)(x_2 - x_3)}(x - x_2), \\ \langle v_5, \xi \rangle &= \frac{r^2}{4} \frac{(\alpha + \beta + (\alpha - \beta)y - 2x_3)}{(x_3 - x_1)(x_3 - x_2)}(x - x_3), \end{aligned} \quad (2.31)$$

and further

$$d\langle B, \xi \rangle \wedge d\langle v_I, \xi \rangle \wedge d\langle v_J, \xi \rangle = (B, v_I, v_J) d\xi^1 \wedge d\xi^2 \wedge d\xi^3 =: \frac{r^4 \rho^2}{16\alpha\beta} K_{IJ} dr^2 \wedge dx \wedge dy, \quad (2.32)$$

$$d\langle v_I, \xi \rangle \wedge d\langle v_J, \xi \rangle \wedge d\langle v_K, \xi \rangle = (v_I, v_J, v_K) d\xi^1 \wedge d\xi^2 \wedge d\xi^3 =: \frac{r^4 \rho^2}{16\alpha\beta} L_{IJK} dr^2 \wedge dx \wedge dy. \quad (2.33)$$

Here $\rho^2 = (1/2)(\alpha + \beta) + (1/2)(\alpha - \beta)y - x$ and (u, v, w) is the determinant of the 3×3 matrix whose rows are equal to the components of u , v and w , respectively. So we have a set of over-determined consistency conditions for α , β and x_i . For example,

$$(v_I, v_J, v_K)L_{I'J'K'} = (v_{I'}, v_{J'}, v_{K'})L_{IJK}, \quad I, J, K, I', J', K' = 1, 2, \dots, 6. \quad (2.34)$$

Some useful relations obtained from (2.34) are

$$\frac{x_1}{x_2} = -\frac{(v_1, v_5, v_6)}{(v_3, v_5, v_6)}, \quad \frac{x_3}{x_2} = -\frac{(v_1, v_5, v_6)}{(v_1, v_3, v_6)}, \quad (2.35)$$

$$\frac{\alpha - x_2}{x_2} = \frac{(v_2, v_3, v_4)}{(v_3, v_4, v_6)}, \quad \frac{\beta - x_2}{x_2} = \frac{(v_2, v_3, v_4)}{(v_2, v_3, v_6)}. \quad (2.36)$$

The constant x_2 can be scaled to be one. Then these relations allow us to represent the set of parameters in terms of the toric data.

The symplectic potential (2.26) satisfies the Monge-Ampère equation and reproduces the $L^{a,b,c}$ metric. See Appendix for details. We note that in the $\alpha = \beta$ limit, the symplectic potential (2.26) goes to the symplectic potential (2.22) for $Y^{p,q}$ up to irrelevant linear terms in ξ^j .

2.5 The suspended pinch point

As an application of the previous subsection, we can explicitly write down the symplectic potential of the suspended pinch point (SPP) model. The toric data is given by [23]

$$v_0 = (1, 0, 0), \quad v_1 = (1, -1, 0), \quad v_2 = (1, 1, 0), \quad v_3 = (1, 1, 1), \quad v_4 = (1, 0, 1). \quad (2.37)$$

Note that $v_1 - v_2 + 2v_3 - 2v_4 = 0$. The subset $\{v_1, v_2, v_3, v_4\}$ is another toric data for $L^{1,2,1}$. In this case, the blow-up vector v_0 is irrelevant for the symplectic potential and one can identify the metric for the SPP model with the $L^{1,2,1}$ metric. Using the vectors $\{v_A\}_{A=1}^4$ for the Z-minimisation, the Reeb vector is determined as [9]

$$B = \left(3, \frac{1}{2}(3 - \sqrt{3}), 3 - \sqrt{3}\right). \quad (2.38)$$

Then, by setting $x_2 = 1$, (2.35) and (2.36) fix the parameters as follows:

$$x_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{3}, \quad x_2 = 1, \quad x_3 = 1 + \sqrt{3}, \quad \alpha = \frac{3}{2} + \frac{1}{2}\sqrt{3}, \quad \beta = \sqrt{3}. \quad (2.39)$$

From (2.28), we have

$$\begin{aligned}\frac{r^2}{2} &= \langle B, \xi \rangle = 3\xi^1 + \frac{1}{2}(3 - \sqrt{3})\xi^2 + (3 - \sqrt{3})\xi^3, \\ x &= \frac{(3/2)(1 + \sqrt{3})\xi^1 + (1/4)(-3 + 2\sqrt{3})\xi^2 + \sqrt{3}\xi^3 - (1/2)(3 - \sqrt{3})\sqrt{D}}{3\xi^1 + (1/2)(3 - \sqrt{3})\xi^2 + (3 - \sqrt{3})\xi^3}, \\ y &= \frac{-(3 + 2\sqrt{3})\xi^1 - (2 + \sqrt{3})\xi^2 - (1 + \sqrt{3})\xi^3 + 2\sqrt{D}}{3\xi^1 + (1/2)(3 - \sqrt{3})\xi^2 + (3 - \sqrt{3})\xi^3},\end{aligned}\quad (2.40)$$

where

$$D = 3(2 + \sqrt{3})(\xi^1)^2 + \frac{1}{8}(14 + 3\sqrt{3})(\xi^2)^2 + 4(\xi^3)^2 + \frac{3}{2}(3 + \sqrt{3})\xi^1\xi^2 + 3(3 + \sqrt{3})\xi^1\xi^3 + 4\xi^2\xi^3. \quad (2.41)$$

By substituting these expressions into (2.26), we obtain the symplectic potential for the SPP model.

3 Scalar Laplacian and Heun's differential equations

In this section, we study the scalar Laplacian for the $L^{a,b,c}$ metric. The eigenvalue equation can be separated into the ordinary differential equations for each variables x, y, ϕ, ψ, τ . The differential equations for the angle variables ϕ, ψ, τ can be solved in a trivial manner. For x and y variables, these are Fuchsian type and are shown to be Heun's differential equations [29].

The $L^{a,b,c}$ metric is given by [3, 4] (see also Appendix)

$$\begin{aligned}ds^2 &= (d\tau + \sigma)^2 + \frac{\rho^2}{4\Delta_x}dx^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 \\ &+ \frac{\Delta_x}{\rho^2} \left(\frac{\sin^2 \theta}{\alpha} d\phi + \frac{\cos^2 \theta}{\beta} d\psi \right)^2 + \frac{\Delta_\theta \sin^2 \theta \cos^2 \theta}{\rho^2} \left(\frac{\alpha - x}{\alpha} d\phi - \frac{\beta - x}{\beta} d\psi \right)^2,\end{aligned}\quad (3.1)$$

where

$$\sigma = \frac{(\alpha - x)}{\alpha} \sin^2 \theta d\phi + \frac{(\beta - x)}{\beta} \cos^2 \theta d\psi, \quad (3.2)$$

$$\Delta_x = x(\alpha - x)(\beta - x) - \mu = (x - x_1)(x - x_2)(x - x_3), \quad (3.3)$$

$$\rho^2 = \Delta_\theta - x, \quad \Delta_\theta = \alpha \cos^2 \theta + \beta \sin^2 \theta. \quad (3.4)$$

Three real roots of Δ_x are chosen such that $0 < x_1 < x_2 < x_3$. It is convenient to change the coordinate θ to y by setting $y = \cos 2\theta$. The coordinates x and y have the ranges $x_1 \leq x \leq x_2$ and $-1 \leq y \leq 1$.

The scalar Laplacian for the $L^{a,b,c}$ metric (3.1) is given by

$$\begin{aligned} \square_{(5)} = & \frac{4}{\rho^2} \frac{\partial}{\partial x} \left(\Delta_x \frac{\partial}{\partial x} \right) + \frac{4}{\rho^2} \frac{\partial}{\partial y} \left(\Delta_y \frac{\partial}{\partial y} \right) + \frac{\partial^2}{\partial \tau^2} \\ & + \frac{\alpha^2 \beta^2}{\rho^2 \Delta_x} \left(\frac{(\beta - x)}{\beta} \frac{\partial}{\partial \phi} + \frac{(\alpha - x)}{\alpha} \frac{\partial}{\partial \psi} - \frac{(\alpha - x)(\beta - x)}{\alpha \beta} \frac{\partial}{\partial \tau} \right)^2 \\ & + \frac{\alpha^2 \beta^2}{\rho^2 \Delta_y} \left(\frac{(1 + y)}{\beta} \frac{\partial}{\partial \phi} - \frac{(1 - y)}{\alpha} \frac{\partial}{\partial \psi} - \frac{(\alpha - \beta)(1 - y^2)}{2\alpha \beta} \frac{\partial}{\partial \tau} \right)^2. \end{aligned} \quad (3.5)$$

Here $\Delta_y := (1 - y^2)\Delta_\theta = (1/2)(1 - y^2)(\alpha + \beta + (\alpha - \beta)y)$. By using (2.30), it also can be written as

$$\begin{aligned} \square_{(5)} = & \frac{\partial^2}{\partial \tau^2} \\ & + \frac{4}{\rho^2} \frac{\partial}{\partial x} \left(\Delta_x \frac{\partial}{\partial x} \right) + \frac{\Delta_x}{\rho^2} \left(\frac{1}{x - x_1} v_1 + \frac{1}{x - x_2} v_3 + \frac{1}{x - x_3} v_5 \right)^2 \\ & + \frac{4}{\rho^2} \frac{\partial}{\partial y} \left(\Delta_y \frac{\partial}{\partial y} \right) + \frac{\Delta_y}{\rho^2} \left(\frac{1}{y - y_1} v_2 + \frac{1}{y - y_2} v_4 + \frac{1}{y - y_3} v_6 \right)^2, \end{aligned} \quad (3.6)$$

where

$$v_1 = -\ell_1, \quad v_3 = -\ell_2, \quad v_5 = -\ell_3. \quad (3.7)$$

$$v_2 = \frac{\partial}{\partial \phi}, \quad v_4 = \frac{\partial}{\partial \psi}, \quad v_6 = \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \psi}. \quad (3.8)$$

Here

$$\ell_i = c_i \frac{\partial}{\partial \tau} + a_i \frac{\partial}{\partial \phi} + b_i \frac{\partial}{\partial \psi}, \quad (3.9)$$

$$a_i = \frac{\alpha c_i}{x_i - \alpha}, \quad b_i = \frac{\beta c_i}{x_i - \beta}, \quad c_i = \frac{(\alpha - x_i)(\beta - x_i)}{2(\alpha + \beta)x_i - \alpha\beta - 3x_i^2}. \quad (3.10)$$

Note that

$$av_1 - cv_2 + bv_3 - dv_4 = 0, \quad v_1 + v_3 + v_5 = B, \quad v_2 + v_4 + v_6 = B. \quad (3.11)$$

Here

$$B = \frac{\partial}{\partial \tau} = B_i \frac{\partial}{\partial \phi_i} \quad (3.12)$$

is the Reeb vector for $L^{a,b,c}$. More explicitly,

$$B = (B_1, B_2, B_3) = \left(3, -\frac{1 + aka_1}{c_1}, -\frac{ba_1}{c_1} \right). \quad (3.13)$$

3.1 Heun's differential equations

The eigenfunctions of the scalar Laplacian, $\square_{(5)}\Psi = -E\Psi$, have the form

$$\Psi = \exp(iN_\tau\tau + iN_\phi\phi + iN_\psi\psi) F(x)G(y). \quad (3.14)$$

Here N_τ, N_ϕ, N_ψ are constants. They are related to constants N^i as $N_\tau\tau + N_\phi\phi + N_\psi\psi = N^i\phi_i$. The coordinates ϕ_i are assumed to be 2π -periodic. Then $N^i \in \mathbb{Z}$ ($i = 1, 2, 3$). It is convenient to introduce the following 1-form:

$$N := N_\phi d\phi + N_\psi d\psi + N_\tau d\tau = N^i d\phi_i. \quad (3.15)$$

For the toric data (2.25), the explicit relations between these constants are given by

$$N_\phi = \langle v_2, N \rangle = N^1 + akN^2 + bN^3, \quad N_\psi = \langle v_4, N \rangle = N^1, \quad N_\tau = \langle B, N \rangle = B_i N^i. \quad (3.16)$$

N_ϕ and N_ψ take values in integers, while N_τ generally takes a value in \mathbb{R} .

The differential equations for F and G can be written as

$$\frac{d^2 F}{dx^2} + \left(\frac{1}{x-x_1} + \frac{1}{x-x_2} + \frac{1}{x-x_3} \right) \frac{dF}{dx} + Q_x F = 0, \quad (3.17)$$

$$\frac{d^2 G}{dy^2} + \left(\frac{1}{y-y_1} + \frac{1}{y-y_2} + \frac{1}{y-y_3} \right) \frac{dG}{dy} + Q_y G = 0, \quad (3.18)$$

where

$$Q_x = \frac{1}{\Delta_x} \left(\mu_x - \frac{1}{4}Ex - \sum_{i=1}^3 \frac{\alpha_i^2 \Delta'_x(x_i)}{x-x_i} \right), \quad Q_y = \frac{1}{H_y} \left(\mu_y - \frac{1}{4}Ey - \sum_{i=1}^3 \frac{\beta_i^2 H'(y_i)}{y-y_i} \right). \quad (3.19)$$

Here

$$\alpha_i := -\frac{1}{2}(a_i N_\phi + b_i N_\psi + c_i N_\tau), \quad (3.20)$$

$$\beta_1 := \frac{1}{2}N_\phi, \quad \beta_2 := \frac{1}{2}N_\psi, \quad \beta_3 := \frac{1}{2}(N_\tau - N_\phi - N_\psi), \quad (3.21)$$

$$H_y := \frac{2\Delta_y}{\beta - \alpha} = (y-y_1)(y-y_2)(y-y_3), \quad (3.22)$$

$$\mu_x := \frac{1}{4}C - \frac{1}{2}N_\tau(\alpha N_\phi + \beta N_\psi) + \frac{1}{4}(\alpha + \beta)N_\tau^2, \quad (3.23)$$

$$\mu_y := \frac{1}{2(\beta - \alpha)} \left(-C + \left(\frac{\alpha + \beta}{2} \right) E + 2(\alpha N_\phi + \beta N_\psi) N_\tau - (\alpha + \beta) N_\tau^2 \right), \quad (3.24)$$

and C is a constant. These differential equations for F and G are the Fuchsian type with four regular singularities at $x = x_1, x_2, x_3, \infty$, and at $y = y_1, y_2, y_3, \infty$, respectively. Therefore, they are Heun's differential equations [29]. For the first Heun's differential equation, the exponents are $\pm\alpha_i$ at $x = x_i$ ($i = 1, 2, 3$), while $-\lambda$ and $\lambda + 2$ at $x = \infty$. For the second, the exponents are $\pm\beta_i$ at $y = y_i$ ($i = 1, 2, 3$), while $-\lambda$ and $\lambda + 2$ at $y = \infty$. Here we put $E = 4\lambda(\lambda + 2)$.

We find that the exponents at $x = x_i$ and $y = y_i$ are related to the toric data as

$$\langle v_1, N \rangle = 2\alpha_1, \quad \langle v_3, N \rangle = 2\alpha_2, \quad \langle v_5, N \rangle = 2\alpha_3, \quad (3.25)$$

$$\langle v_2, N \rangle = 2\beta_1, \quad \langle v_4, N \rangle = 2\beta_2, \quad \langle v_6, N \rangle = 2\beta_3. \quad (3.26)$$

These are important relations which will be useful to discuss properties of the eigenfunctions. Note that from (3.20) and (3.21), $\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3 = (1/2)N_\tau$.

Heun's differential equations (3.17) and (3.18) can be converted into the standard forms by the coordinate transformations

$$\tilde{x} = \frac{x - x_1}{x_2 - x_1}, \quad \tilde{y} = \frac{y - y_1}{y_2 - y_1}, \quad (3.27)$$

with the rescalings³

$$F(x) = \tilde{x}^{\alpha_1} (1 - \tilde{x})^{\alpha_2} (a_x - \tilde{x})^{\alpha_3} f(\tilde{x}), \quad a_x := \frac{x_3 - x_1}{x_2 - x_1}, \quad (3.28)$$

$$G(y) = \tilde{y}^{\beta_1} (1 - \tilde{y})^{\beta_2} (a_y - \tilde{y})^{\beta_3} g(\tilde{y}), \quad a_y := \frac{y_3 - y_1}{y_2 - y_1}. \quad (3.29)$$

The resulting two standard form of Heun's differential equations are

$$\frac{d^2 f}{d\tilde{x}^2} + \left(\frac{\gamma_x}{\tilde{x}} + \frac{\delta_x}{\tilde{x} - 1} + \frac{\epsilon_x}{\tilde{x} - a_x} \right) \frac{df}{d\tilde{x}} + \frac{\hat{\alpha}\hat{\beta}\tilde{x} - k_x}{\tilde{x}(\tilde{x} - 1)(\tilde{x} - a_x)} f = 0, \quad (3.30)$$

$$\frac{d^2 g}{d\tilde{y}^2} + \left(\frac{\gamma_y}{\tilde{y}} + \frac{\delta_y}{\tilde{y} - 1} + \frac{\epsilon_y}{\tilde{y} - a_y} \right) \frac{dg}{d\tilde{y}} + \frac{\hat{\alpha}\hat{\beta}\tilde{y} - k_y}{\tilde{y}(\tilde{y} - 1)(\tilde{y} - a_y)} g = 0, \quad (3.31)$$

³Here, for simplicity, we assume that $\alpha_1, \alpha_2, \beta_1$, and β_2 are all positive. All negative case can be treated by replacing the powers in (3.28) and (3.29): $\alpha_i \rightarrow -\alpha_i, \beta_i \rightarrow -\beta_i$. Otherwise various relations between the constants a_i, b_i , and c_i (3.10) cannot be used and the analysis becomes very complicated. We will comment on this assumption later. See section 5.

where

$$\hat{\alpha} = -\lambda + \frac{1}{2}N_\tau, \quad \hat{\beta} = 2 + \lambda + \frac{1}{2}N_\tau. \quad (3.32)$$

$$\gamma_x = 2\alpha_1 + 1, \quad \delta_x = 2\alpha_2 + 1, \quad \epsilon_x = 2\alpha_3 + 1, \quad (3.33)$$

$$\gamma_y = 2\beta_1 + 1, \quad \delta_y = 2\beta_2 + 1, \quad \epsilon_y = 2\beta_3 + 1, \quad (3.34)$$

and the accessory parameters are given by

$$k_x = (\alpha_1 + \alpha_3)(\alpha_1 + \alpha_3 + 1) - \alpha_2^2 + a_x[(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + 1) - \alpha_3^2] - \tilde{\mu}_x, \quad (3.35)$$

$$k_y = (\beta_1 + \beta_3)(\beta_1 + \beta_3 + 1) - \beta_2^2 + a_y[(\beta_1 + \beta_2)(\beta_1 + \beta_2 + 1) - \beta_3^2] - \tilde{\mu}_y. \quad (3.36)$$

Here

$$\tilde{\mu}_x := \frac{1}{x_2 - x_1} \left(\mu_x - \frac{1}{4}Ex_1 \right), \quad \tilde{\mu}_y := \frac{1}{y_2 - y_1} \left(\mu_y - \frac{1}{4}Ey_1 \right). \quad (3.37)$$

Note that the parameters $\hat{\alpha}$ and $\hat{\beta}$ are common for both Heun's differential equations. With some work, the accessory parameters are summarised as follows:

$$k_x = \frac{1}{x_2 - x_1}(\tilde{C} - \hat{\alpha}\hat{\beta}x_1), \quad k_y = \frac{1}{\beta - \alpha}(\tilde{C} - \hat{\alpha}\hat{\beta}\alpha). \quad (3.38)$$

Here

$$\tilde{C} := \frac{1}{2}(\alpha + \beta)N_\tau - \frac{1}{2}(\alpha N_\phi + \beta N_\psi) - \frac{1}{4}C. \quad (3.39)$$

In the following, \tilde{C} will be treated as an undetermined constant.

3.2 Comments on the $Y^{p,q}$ limit

For $\alpha = \beta$ limit, the $L^{a,b,c}$ metric reduces to the $Y^{p,q}$ metric. In this limit, y_3 goes to ∞ and the differential equation for the y -system (3.18) becomes

$$\frac{d^2G}{dy^2} + \left(\frac{1}{y-1} + \frac{1}{y+1} \right) \frac{dG}{dy} + \sum_{i=1}^2 \left(-\frac{\beta_i^2}{(y-y_i)^2} + \frac{Q_i^{(1)}}{y-y_i} \right) G = 0, \quad (3.40)$$

where

$$Q_1^{(1)} = -Q_2^{(1)} = \frac{1}{8}(\alpha^{-1}C - E + N_\tau^2 - 2N_\phi N_\psi). \quad (3.41)$$

This differential equation is the Fuchsian type with three regular singularities at $\pm 1, \infty$. The exponents are $\pm\beta_i$ at $y = y_i$ ($i = 1, 2$) and $-J, J+1$ at $y = \infty$. Here J is defined by

$$J(J+1) = \beta_1^2 + \beta_2^2 - 2Q_1^{(1)} = \frac{1}{4} [E - \alpha^{-1}C - (N_\tau - N_\phi - N_\psi)(N_\tau + N_\phi + N_\psi)]. \quad (3.42)$$

One can solve the differential equation in terms of Gauss' hypergeometric function:

$$G(y) = (y - y_1)^{\beta_1} (y - y_2)^{\beta_2} F(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}; (1/2)(1 - y)), \quad (3.43)$$

where

$$\tilde{\alpha} = \beta_1 + \beta_2 - J, \quad \tilde{\beta} = \beta_1 + \beta_2 + J + 1, \quad \tilde{\gamma} = 1 + 2\beta_1. \quad (3.44)$$

When $\tilde{\alpha}$ is a non-positive integer, the hypergeometric function becomes a Jacobi polynomial. The constant J here is the exact $SU(2)$ -spin J in [32].

4 Polynomial solutions

In this section, the spectrum of the scalar Laplacian (3.5) of $L^{a,b,c}$ is investigated. The $L^{a,b,c}$ family contains $T^{1,1}$ and $Y^{p,q}$ as special cases. The case of $T^{1,1}$ was studied in [33] and the case of $Y^{p,q}$ in [11, 32].

The spectrum can be determined by solving the two Heun's differential equations. In principle, one can obtain local power-series solutions of Heun's by solving three-term recursion relations. But it is not easy to write the power series in a compact manner. Also, the local solutions should be regular functions at the regular singularities 0 and 1. Such regular solutions are called the Heun functions [29]. The regular solutions of polynomial type appear when the three-term recursion relations terminate at certain degree. We study the polynomial solutions for the two Heun's differential equations.

4.1 Recursion relations

For $\lambda = \frac{1}{2}N_\tau + n$ ($n \in \mathbb{Z}_{\geq 0}$), the parameters $\hat{\alpha}$ and $\hat{\beta}$ take values as $\hat{\alpha} = -n$, $\hat{\beta} = N_\tau + n + 2$, and a polynomial solution of degree n may be possible:

$$f(\tilde{x}) = \sum_{m=0}^n d_m \tilde{x}^m, \quad g(\tilde{y}) = \sum_{m=0}^n e_m \tilde{y}^m. \quad (4.1)$$

Since the parameters $\hat{\alpha}$ and $\hat{\beta}$ are common for both Heun's differential equations, the degree of both polynomials must be equal, if there exist.

The coefficients d_m and e_m satisfy the following three-term recursion relations:

$$\begin{aligned} -k_x d_0 + a_x \gamma_x d_1 &= 0, \\ P_m d_{m-1} - (Q_m^{(x)} + k_x) d_m + R_m^{(x)} d_{m+1} &= 0, \quad (m = 1, 2, \dots, n-1), \\ P_n d_{n-1} - (Q_n^{(x)} + k_x) d_n &= 0. \end{aligned} \quad (4.2)$$

$$\begin{aligned} -k_y e_0 + a_y \gamma_y e_1 &= 0, \\ P_m e_{m-1} - (Q_m^{(y)} + k_y) e_m + R_m^{(y)} e_{m+1} &= 0, \quad (m = 1, 2, \dots, n-1), \\ P_n e_{n-1} - (Q_n^{(y)} + k_y) e_n &= 0. \end{aligned} \quad (4.3)$$

Here

$$\begin{aligned} P_m &= (m-1 + \hat{\alpha})(m-1 + \hat{\beta}), \\ Q_m^{(x)} &= m((m-1 + \gamma_x)(1 + a_x) + a_x \delta_x + \epsilon_x), \\ Q_m^{(y)} &= m((m-1 + \gamma_y)(1 + a_y) + a_y \delta_y + \epsilon_y), \\ R_m^{(x)} &= (m+1)(m + \gamma_x) a_x, \\ R_m^{(y)} &= (m+1)(m + \gamma_y) a_y. \end{aligned} \quad (4.4)$$

The requirement for existence of nontrivial solutions can be written in the following form:

$$\det(M_x^{(n)} - k_x) = 0, \quad \det(M_y^{(n)} - k_y) = 0. \quad (4.5)$$

Here $M_x^{(n)}$ and $M_y^{(n)}$ are $(n+1) \times (n+1)$ matrices whose matrix elements can be read from (4.2) and (4.3) respectively. These relations (4.5) are two algebraic equations for one constant \tilde{C} which is hidden in the accessory parameters k_x and k_y . They impose quite strong restrictions on \tilde{C} .

4.2 Constant solutions ($n = 0$)

For $n = 0$, let us set $\tilde{C} = 0$. Then $k_x = k_y = 0$ and we have constant solutions with the eigenvalues

$$E = N_\tau(N_\tau + 4). \quad (4.6)$$

The constant solutions obtained here are trivial ones to Heun's differential equations. However, due to the rescaling factors in (3.28) and (3.29), these correspond to nontrivial eigenfunctions of the scalar Laplacian. These are closely related to holomorphic functions on the cone $C(X_5)$. In the next section, we will discuss this point in detail.

4.3 $n = 1$

Next, let us examine the first excited states ($n = 1$). The polynomial conditions (4.5) for this case

$$k_x(k_x + Q_1^{(x)}) - a_x \gamma_x P_1 = 0, \quad k_y(k_y + Q_1^{(y)}) - a_y \gamma_y P_1 = 0, \quad (4.7)$$

yield the same algebraic equation for \tilde{C} :

$$\tilde{C}^2 + 2\nu\tilde{C} + \alpha\beta(N_\tau + 3)(N_\tau - N_\phi - N_\psi + 1) = 0, \quad (4.8)$$

where

$$\nu := \frac{1}{2}\alpha(N_\tau + 2 - N_\phi) + \frac{1}{2}\beta(N_\tau + 2 - N_\psi). \quad (4.9)$$

Therefore, if

$$\tilde{C} = -\nu \pm \sqrt{\nu^2 - \alpha\beta(N_\tau + 3)(N_\tau - N_\phi - N_\psi + 1)}, \quad (4.10)$$

then there exist polynomial solutions of degree one for both x and y systems. The eigenvalues of the scalar Laplacian are given by

$$E = (N_\tau + 2)(N_\tau + 6). \quad (4.11)$$

4.4 Comments for $n \geq 2$

We comment on the cases for $n \geq 2$. For $n = 2$, we can show that

$$(x_1 - x_2)^3 \det(M_x^{(2)} - k_x) - (\alpha - \beta)^3 \det(M_y^{(2)} - k_y) = 4\mu(N_\tau + 4)(N_\tau + 5). \quad (4.12)$$

Therefore, $\det(M_x^{(2)} - k_x)$ and $\det(M_y^{(2)} - k_y)$ cannot vanish simultaneously, and so the polynomial solutions for both x and y systems are not allowed in this case.

For $n = 3$, the following relation holds

$$\begin{aligned} & (x_1 - x_2)^4 \det(M_x^{(3)} - k_x) - (\alpha - \beta)^4 \det(M_y^{(3)} - k_y) \\ &= 12\mu(N_\tau + 6)(2(N_\tau + 6)\tilde{C} + 3(\alpha + \beta)(N_\tau + 4)(N_\tau + 5) - 3(N_\tau + 5)(\alpha N_\phi + \beta N_\psi)). \end{aligned} \quad (4.13)$$

If there is a constant \tilde{C} such that both $\det(M_x^{(3)} - k_x)$ and $\det(M_y^{(3)} - k_y)$ vanish, the right-handed side of the equation above must be zero. But it seems that

$$\tilde{C} = -\frac{3(N_\tau + 5)}{2(N_\tau + 6)}((\alpha + \beta)(N_\tau + 4) - (\alpha N_\phi + \beta N_\psi)) \quad (4.14)$$

is not a solution of $\det(M_x^{(3)} - k_x) = 0$ or $\det(M_y^{(3)} - k_y) = 0$. So there is no simultaneous polynomial solution.

In general, for $n \geq 2$, $\det(M_x^{(n)} - k_x) = 0$ and $\det(M_y^{(n)} - k_y) = 0$ give two different algebraic equations for one constant \tilde{C} . It seems that they do not have a common solution and hence there is no simultaneous polynomial solution of degree n for both x and y systems. We have to search for the Heun functions of non-polynomial type at least for one Heun's.

5 Holomorphic functions and BPS mesons

In subsection 4.2, we determined the constant solutions to Heun's differential equations. Up to a normalisation constant, the corresponding eigenfunctions are summarised as

$$\Psi^{(0)}[N] = e^{iN^i \phi_i} (x - x_1)^{\alpha_1} (x_2 - x)^{\alpha_2} (x_3 - x)^{\alpha_3} (1 - y)^{\beta_1} (1 + y)^{\beta_2} (1 - (y/y_3))^{\beta_3}. \quad (5.1)$$

In deriving the solutions above, we have assumed that

$$\alpha_1 \geq 0, \quad \alpha_2 \geq 0, \quad \beta_1 \geq 0, \quad \beta_2 \geq 0. \quad (5.2)$$

With this assumption, the ground states (5.1) are regular in the regions $x_1 \leq x \leq x_2$, $-1 \leq y \leq 1$. Note that (5.2) are sufficient conditions for the eigenfunctions (5.1) of the scalar Laplacian being regular.

Anyway, let us discuss the relation between the solution (5.1) and holomorphic functions. Since $C(X_5)$ is a Kähler cone, any holomorphic function w satisfies the Laplace equation $\square_{(6)} w = 0$. By restricting to the base space X_5 , the holomorphic function with the scaling dimension Δ becomes an eigenfunction of $\square_{(5)}$ with eigenvalue $E = \Delta(\Delta + 4)$. The scaling dimension of the holomorphic function is defined by $r(\partial/\partial r)w = \Delta w$.

In the symplectic approach to the Calabi-Yau cones, there is a way to construct the holomorphic coordinates by using a symplectic potential G . The holomorphic coordinates are given by [9]

$$w_i := \text{const} \times \exp \left(\frac{\partial G(\xi)}{\partial \xi^i} + i\phi_i \right), \quad i = 1, 2, 3. \quad (5.3)$$

The coordinates w_i are periodic functions under the shift $\phi_i \rightarrow \phi_i + 2\pi$. In these holomorphic coordinates, the holomorphic $(3, 0)$ -form Ω is given by $\Omega = dw_1 \wedge dw_2 \wedge dw_3 / (w_2 w_3)$.

Since we know the symplectic potential (2.26), we can write down the explicit form of the holomorphic coordinates on $C(L^{a,b,c})$. From (A.2), and with an appropriate choice of

the normalisation, we have

$$w_i = r^{B_i} e^{i\phi_i} (x - x_1)^{(1/2)(v_1)_i} (x_2 - x)^{(1/2)(v_3)_i} (x_3 - x)^{(1/2)(v_5)_i} \\ \times (1 - y)^{(1/2)(v_2)_i} (1 + y)^{(1/2)(v_4)_i} \left(1 - \frac{y}{y_3}\right)^{(1/2)(v_6)_i}. \quad (5.4)$$

The scaling dimension of w_i is equal to the i -th component of the Reeb vector B , and so the dimension Δ_N of a holomorphic function $\Psi[N] := w_1^{N^1} w_2^{N^2} w_3^{N^3}$ is given by

$$\Delta_N = \langle B, N \rangle = B_i N^i = N_\tau. \quad (5.5)$$

Recalling (2.28) and (2.31), we see that the facets $\langle v_A, \xi \rangle = 0$ ($A = 1, 2, 3, 4$) correspond to $x = x_1$, $y = 1$, $x = x_2$ and $y = -1$, respectively. The necessary condition for the regularity of $\Psi[N]$ at the facets requires

$$\langle v_A, N \rangle \geq 0, \quad A = 1, 2, 3, 4. \quad (5.6)$$

Therefore, there exists a holomorphic function to each integral lattice point (N^1, N^2, N^3) in the polyhedral cone. This fact is well-known in toric geometry. Using (3.25), (3.26) and (3.15), we can see that the restriction gives the eigenfunction (5.1),

$$\Psi^{(0)}[N] = w_1^{N^1} w_2^{N^2} w_3^{N^3} \Big|_{r=1}. \quad (5.7)$$

The eigenvalue is given by $E = \Delta_N(\Delta_N + 4) = N_\tau(N_\tau + 4)$, which is consistent with (4.6).

The AdS/CFT implies that these eigenfunctions correspond to the BPS mesonic operators with the R -charge $(2/3)\Delta_N$. For general $L^{a,b,c}$, it is difficult to treat all integral points in the polyhedral cone.

One obvious integral point is $(N^1, N^2, N^3) = (1, 0, 0) =: u_0$. This vector lies in the polyhedral cone for any toric data. Therefore, it is natural to identify its dual with the short BPS meson operator \mathcal{O}_β with the R -charge 2, which exists for any toric superconformal quiver gauge theory [34, 16]. The eigenfunction is given by

$$\Psi^{(0)}[u_0] = e^{i\phi_1} \prod_{m=1}^3 (x - x_m)^{1/2} \left(1 - \frac{y}{y_m}\right)^{1/2}. \quad (5.8)$$

Other manifest possibilities are four vectors pointing at the directions along the four edges of the polyhedral cone:

$$\begin{aligned} u_1 &:= v_1 \times v_2 = (b, -b, ak - 1), \\ u_2 &:= v_2 \times v_3 = (a, b - c, -a(k + l)), \\ u_3 &:= v_3 \times v_4 = (0, c, al), \\ u_4 &:= v_4 \times v_1 = (0, 0, 1). \end{aligned} \quad (5.9)$$

Note that they obey a linear relation:

$$acu_1 - bcu_2 + bdu_3 - adu_4 = 0. \quad (5.10)$$

Here we assume that the vectors u_i are primitive, i.e., whose components do not have a common factor. If these vectors are not primitive, which are related to the orbifolds of the Sasaki-Einstein manifolds, more careful treatment is needed. But we do not argue this subtle point. The corresponding eigenfunctions are explicitly given by

$$\begin{aligned} \Psi^{(0)}[u_1] &= e^{ib\phi_1 - ib\phi_2 + i(ak-1)\phi_3} \\ &\quad \times (x_2 - x)^{(1/2)d} (x_3 - x)^{(1/2)\langle v_5, u_1 \rangle} (1 + y)^{(1/2)b} (1 - (y/y_3))^{(1/2)\langle v_6, u_1 \rangle}, \\ \Psi^{(0)}[u_2] &= e^{ia\phi_1 + i(b-c)\phi_2 - ia(k+l)\phi_3} \\ &\quad \times (x - x_1)^{(1/2)d} (x_3 - x)^{(1/2)\langle v_5, u_2 \rangle} (1 + y)^{(1/2)a} (1 - (y/y_3))^{(1/2)\langle v_6, u_2 \rangle}, \\ \Psi^{(0)}[u_3] &= e^{ic\phi_2 + ial\phi_3} \\ &\quad \times (x - x_1)^{(1/2)c} (x_3 - x)^{(1/2)\langle v_5, u_3 \rangle} (1 - y)^{(1/2)a} (1 - (y/y_3))^{(1/2)\langle v_6, u_3 \rangle}, \\ \Psi^{(0)}[u_4] &= e^{i\phi_3} \\ &\quad \times (x_2 - x)^{(1/2)c} (x_3 - x)^{(1/2)\langle v_5, u_4 \rangle} (1 - y)^{(1/2)b} (1 - (y/y_3))^{(1/2)\langle v_6, u_4 \rangle}. \end{aligned} \quad (5.11)$$

It is natural to identify primary operators related to these four integral points u_i ($i = 1, 2, 3, 4$) with the four extremal BPS meson operators \mathcal{O}_{LD} , \mathcal{O}_{RD} , \mathcal{O}_{RU} and \mathcal{O}_{LU} , respectively [16]. They also correspond to four (p, q) -branes [16, 17], whose (p, q) -charges can be read from the second and the third components of u_i : $(p, q) = (-(u_i)^2, -(u_i)^3)$. Classical counterparts are the four massless BPS geodesics considered in [16]⁴. Using the notation of [17] for the fundamental chiral fields, the R -charges of the extremal BPS mesons are related to the toric data as follows:

$$\begin{aligned} R[\mathcal{O}_{LD}] &= dR[Z] + bR[U_2] = \frac{2}{3}\langle B, v_1 \times v_2 \rangle, & R[\mathcal{O}_{RD}] &= dR[Y] + aR[U_2] = \frac{2}{3}\langle B, v_2 \times v_3 \rangle, \\ R[\mathcal{O}_{RU}] &= cR[Y] + aR[U_1] = \frac{2}{3}\langle B, v_3 \times v_4 \rangle, & R[\mathcal{O}_{LU}] &= cR[Z] + bR[U_1] = \frac{2}{3}\langle B, v_4 \times v_1 \rangle. \end{aligned} \quad (5.12)$$

By comparing the powers of $(x - x_1)^{1/2}$, $(x_2 - x)^{1/2}$, $(1 - y)^{1/2}$, $(1 + y)^{1/2}$ in (5.11) with the coefficients of $R[Y]$, $R[Z]$, $R[U_1]$, $R[U_2]$ in (5.12), we find the relation between the Reeb vector and the R -charges of the distinguished bifundamental fields Y , U_1 , Z , U_2 :

$$\frac{2}{3}B = R[Y]v_1 + R[U_1]v_2 + R[Z]v_3 + R[U_2]v_4. \quad (5.13)$$

⁴The conditions $y = \pm 1$ in our notation seem to correspond to the positions $y = \mp 1$ where the massless BPS geodesics stay [16].

The Reeb vector B can be determined by the Z -minimisation, while the R -charges can be computed by the α -maximisation [35]. This relation connects the charges in the gauge theory to the geometric object B , and is a consequence of the AdS/CFT correspondence.

6 Summary and Discussion

In this paper, we have presented the symplectic potentials for a wide class of toric Sasaki-Einstein manifolds.

The spectrum of the scalar Laplacian for $L^{a,b,c}$ is investigated. The eigenvalue problem leads to two Heun's differential equations, which are correlated through one constant \tilde{C} . We find that the exponents at the regular singularities can be nicely characterised by the toric data.

What is the physical interpretation of \tilde{C} ? The eigenvalue equations are separated into five ordinary differential equations. There are four manifestly constant quantities. Three are related to the three $U(1)$ isometries. The fourth is E , which is the “energy.” The physical meaning of these four constants are clear. The fifth constant \tilde{C} appears through the separation of variables for x and y . \tilde{C} is not a Noether charge. Originally, the $L^{a,b,c}$ metrics were reduced from the Kerr-AdS black holes [3, 4]. This way of generating Sasaki-Einstein manifolds was started out with [36], where the special case $Y^{p,q}$ was studied. The integrability property of geodesics and eigenvalue equations on the black holes is guaranteed by the existence of a symmetric rank two Killing tensor [37]. Therefore, there may be a connection between the constant \tilde{C} and the Killing tensor.

The ground states can be obtained as a restriction of the holomorphic functions on the Calabi-Yau cone. Some families of holomorphic functions were constructed for $Y^{p,q}$ [11]. But the analysis was done in rather ad hoc manner. By combining knowledge of the explicit symplectic potential (2.26) and the relationship between the exponents and the toric data (3.25), (3.26), we have shown that the eigenfunctions (5.1) or the holomorphic functions $w_1^{N^1} w_2^{N^2} w_3^{N^3}$ have one-to-one correspondence with integral points (N^1, N^2, N^3) in the convex polyhedral cone. The holomorphic functions have the scaling dimensions $B_i N^i$. The corresponding values of R -charges $(2/3)B_i N^i$ are consistent with the results of the dual quiver gauge theories. For certain BPS operators, the corresponding integers $(-N^2, -N^3)$ are found to be equal to the (p, q) -charges of the (p, q) -web of 5-branes [38]. Moreover, the powers of $(x - x_1)^{1/2}$, $(x_2 - x)^{1/2}$, $(1 - y)^{1/2}$, $(1 + y)^{1/2}$ are closely related to the numbers of the “constituent” bifundamental fields Y , Z , U_1 , U_2 , respectively. So there may be more physical explanation of these eigenfunctions.

In addition to the ground states, we constructed the first excited eigenfunctions. Due to the difficulties for obtaining the Heun functions, the determination of the full spectrum for $L^{a,b,c}$ is still an open problem.

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A Details on the symplectic potential for $L^{a,b,c}$

In this appendix, we show that the symplectic potential (2.26) is indeed a solution to the Monge-Ampère equation for $L^{a,b,c}$. The key relations, obtained from (2.28), (2.29) and (2.31), are

$$\frac{\langle v_1, \xi \rangle}{x - x_1} + \frac{\langle v_3, \xi \rangle}{x - x_2} + \frac{\langle v_5, \xi \rangle}{x - x_3} = 0, \quad \frac{\langle v_2, \xi \rangle}{y - 1} + \frac{\langle v_4, \xi \rangle}{y + 1} + \frac{\langle v_6, \xi \rangle}{y - y_3} = 0. \quad (\text{A.1})$$

Using these relations, we have (up to an irrelevant constant)

$$\begin{aligned} \frac{\partial G}{\partial \xi^i} &= \frac{1}{2} B_i \log \langle B, \xi \rangle \\ &+ \frac{1}{2} (v_1)_i \log |x - x_1| + \frac{1}{2} (v_3)_i \log |x - x_2| + \frac{1}{2} (v_5)_i \log |x - x_3| \\ &+ \frac{1}{2} (v_2)_i \log |y - 1| + \frac{1}{2} (v_4)_i \log |y + 1| + \frac{1}{2} (v_6)_i \log |y - y_3|. \end{aligned} \quad (\text{A.2})$$

Recall that $B_1 = 3$, and $(v_I)_1 = 1$, for $I = 1, 2, \dots, 6$. Then (A.2) for $i = 1$ leads to

$$2 \frac{\partial G}{\partial \xi^1} = \log \left| \langle B, \xi \rangle^3 (x - x_1)(x - x_2)(x - x_3)(1 - y^2)(y - y_3) \right|. \quad (\text{A.3})$$

By differentiating (A.2) once more, we have

$$\begin{aligned} \frac{\partial^2 G}{\partial \xi^i \partial \xi^j} &= \frac{B_i B_j}{2 \langle B, \xi \rangle} + \frac{1}{2} \left[\frac{(v_1)_i}{x - x_1} + \frac{(v_3)_i}{x - x_2} + \frac{(v_5)_i}{x - x_3} \right] \frac{\partial x}{\partial \xi^j} \\ &+ \frac{1}{2} \left[\frac{(v_2)_i}{y - 1} + \frac{(v_4)_i}{y + 1} + \frac{(v_6)_i}{y - y_3} \right] \frac{\partial y}{\partial \xi^j}. \end{aligned} \quad (\text{A.4})$$

Contraction with $d\xi^i d\xi^j$ yields

$$\begin{aligned} \frac{\partial^2 G}{\partial \xi^i \partial \xi^j} d\xi^i d\xi^j &= \frac{(d\langle B, \xi \rangle)^2}{2\langle B, \xi \rangle} + \frac{1}{2} \left[\frac{d\langle v_1, \xi \rangle}{x - x_1} + \frac{d\langle v_3, \xi \rangle}{x - x_2} + \frac{d\langle v_5, \xi \rangle}{x - x_3} \right] dx \\ &\quad + \frac{1}{2} \left[\frac{d\langle v_2, \xi \rangle}{y - 1} + \frac{d\langle v_4, \xi \rangle}{y + 1} + \frac{d\langle v_6, \xi \rangle}{y - y_3} \right] dy. \end{aligned} \quad (\text{A.5})$$

From (A.1), the following relations can be obtained

$$\frac{d\langle v_1, \xi \rangle}{x - x_1} + \frac{d\langle v_3, \xi \rangle}{x - x_2} + \frac{d\langle v_5, \xi \rangle}{x - x_3} = \left[\frac{\langle v_1, \xi \rangle}{(x - x_1)^2} + \frac{\langle v_3, \xi \rangle}{(x - x_2)^2} + \frac{\langle v_5, \xi \rangle}{(x - x_3)^2} \right] dx = \frac{r^2 \rho^2}{2\Delta_x} dx, \quad (\text{A.6})$$

$$\frac{d\langle v_2, \xi \rangle}{y - 1} + \frac{d\langle v_4, \xi \rangle}{y + 1} + \frac{d\langle v_6, \xi \rangle}{y - y_3} = \left[\frac{\langle v_2, \xi \rangle}{(y - 1)^2} + \frac{\langle v_4, \xi \rangle}{(y + 1)^2} + \frac{\langle v_6, \xi \rangle}{(y - y_3)^2} \right] dy = \frac{r^2 \rho^2}{2\Delta_y} dy. \quad (\text{A.7})$$

Here

$$\Delta_x = (x - x_1)(x - x_2)(x - x_3), \quad \Delta_y = \frac{1}{2}(1 - y^2)((\alpha + \beta) + (\alpha - \beta)y). \quad (\text{A.8})$$

Substitution of (A.6) and (A.7) into (A.5) yields

$$\frac{\partial^2 G}{\partial \xi^i \partial \xi^j} d\xi^i d\xi^j = dr^2 + \frac{r^2 \rho^2}{4\Delta_x} dx^2 + \frac{r^2 \rho^2}{4\Delta_y} dy^2. \quad (\text{A.9})$$

Note that the Jacobian of the coordinate transformation (2.28) from (ξ^1, ξ^2, ξ^3) to (r, x, y) is proportional to $r^5 \rho^2$. From (A.9) and the Jacobian, one can show that

$$\det \left(\frac{\partial^2 G}{\partial \xi^i \partial \xi^j} \right) \propto \frac{1}{r^6 \Delta_x \Delta_y} \propto \frac{1}{\langle B, \xi \rangle^3 (x - x_1)(x - x_2)(x - x_3)(1 - y^2)(y - y_3)}. \quad (\text{A.10})$$

Hence, with (A.3), we have

$$\det \left(\frac{\partial^2 G}{\partial \xi^i \partial \xi^j} \right) = \text{const} \times \exp \left(-2 \frac{\partial G}{\partial \xi^1} \right), \quad (\text{A.11})$$

which implies that (2.26) is a solution to the Monge-Ampère equation. Moreover, (A.9) implies that the symplectic potential (2.26) indeed reproduces the $L^{a,b,c}$ metric (3.1) by setting $y = \cos 2\theta$.

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